PROPAGATION OF SMALL PERTURBATIONS IN A LIQUID WITH BUBBLES

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The problem of the propagation of small perturbations in a liquid with bubbles has been studied in one form or another in [1-7] and for a gas with particles in [8].

1. Linearization of the System of Equations. We consider the propagation of small perturbations in a liquid with gas bubbles under the following assumptions. The wavelength of sound is very much longer than the average distance between bubbles, which is very much larger than the size of the bubbles; i.e., the cubic content of the gas phase α is rather small, $\alpha < 0.1$. The mixture is polydisperse; i.e., in each elementary volume there are $m - 1$ kinds of bubbles, all containing the same gas. Capillary effects are neglected (the bubbles are not very small). The viscosity and thermal conductivity are important only in interactions between bubbles and the liquid phase, and in radial pulsations.

We use the linearized system of equations [9] generalized for the polydisperse case. The equations of conservation of mass, the number of bubbles, momentum, energy, and pulsational motion have the following form:

$$\begin{gathered}\frac{\partial\rho_1}{\partial t}+\rho_{10}\operatorname{div}v_1=-\sum_{i=2}^{m}I_i,\quad \frac{\partial\rho_i}{\partial t}+\rho_{10}\operatorname{div}v_i=I_i,\quad \frac{\partial n_i}{\partial t}+n_{i0}\operatorname{div}v_i=0,\\ \rho_{10}\frac{\partial v_1}{\partial t}=-\alpha_{10}\nabla p_1-\sum_{i=2}^{m}f_i,\quad \rho_{i0}\frac{\partial v_i}{\partial t}=-\alpha_{i0}\nabla p_1+f_i,\\ \rho_{10}\frac{\partial u_1}{\partial t}=\frac{\alpha_{10}p_0}{\rho_{10}^0}\frac{\partial\rho_1^0}{\partial t}-\sum_{i=2}^{m}q_i^0,\quad \rho_{i0}\frac{\partial u_i}{\partial t}=\frac{\alpha_{i0}p_0}{\rho_{i0}^0}\frac{\partial\rho_i^0}{\partial t}-q_{0i},\quad q_i^0+q_{0i}=lI_i,\\ \frac{1}{2}\frac{\partial\delta_i}{\partial t}=w_i+\frac{I_i}{\pi\delta_{i0}^2 n_{i0}\rho_{i0}^0},\quad \frac{\delta_{i0}}{2}\frac{\partial w_i}{\partial t}=\frac{p_i-p_1}{\rho_{10}^0}-8\frac{\nu_1}{\delta_{i0}}w_i,\end{gathered}\tag{1.1}$$

where ρ, ρ^0, v, p, n, u, w, and δ are, respectively, the perturbations of the average density, true density, velocity, pressure, number of bubbles per unit volume, internal energy, radial mass velocity of the liquid at the phase boundary, and diameter of the bubbles; the α_i are the volume concentrations of the phases; l is the heat of vaporization; and ν_1 is the kinematic viscosity of the liquid.

The system of equations (1.1) will be closed if we specify expressions for the interaction force f_i, heat transfer q_i^0, q_{0i}, mass transfer I_i, the equations of state of the phases, and certain kinematic relations.

The following relations can be taken for the interaction force and heat transfer:

$$\begin{gathered}f_i=f_{mi}+f_{\mu i},\quad f_{mi}=\frac{1}{2}\frac{\pi n_{i0}\delta_{i0}}{6}\rho_{10}^0\frac{\partial}{\partial t}(v_1-v_i),\quad f_{\mu i}=\chi n_{i0}\delta_{i0}\nu_1\rho_{10}^0(v_1-v_i);\\ q_i^0=\pi n_{i0}\delta_{i0}\mathrm{Nu}_i^0(T_1-T_{\sigma i}),\quad q_{0i}=\pi n_{i0}\delta_{i0}\mathrm{Nu}_{0i}(T_i-T_{\sigma i}),\end{gathered}$$

where $\chi = 3\pi$ for Stokes' law of flow and $\chi = 6\pi$ for Levich's law [10]; Nu_i^0 and Nu_{0i} are the Nusselt numbers for surface heat transfer from the liquid and gas phases, respectively.

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The equation for the phase-transition kinetics for the linear nonequilibrium theory can be written in the form

$$I_i = \pi n_{i0}\delta_{i0}^2\beta_i(T_{\sigma i} - T_{si}) = \pi n_{i0}\delta_{i0}^2 \frac{l_0}{T_0} F_i \frac{(T_{\sigma i} - T_{\rho i})}{T_0}, \left(F_i = \frac{\beta_i T_0^2}{l_0}\right),$$

where β_i is the mass-transfer coefficient. If it is assumed that $T_{\sigma i} = T_{si}$, i.e., that the temperature at the surface of the bubbles is equal to the saturation temperature at the pressure inside the bubbles (quasiequilibrium or equilibrium model [11]), the intensity of the phase transitions is determined automatically from the energy equation of the surface phase.

As the equations of state of a two-phase single-component system we use the relations

$$p_1 = p_0 + a_1^2(\rho_1^0 - \rho_{10}^0), \quad u_1 = c_1(T_1 - T_0) - \int_{p_0}^{p_1} p_1 d(1/\rho_1^0),$$

$$p_i = \rho_i^0 R_2 T_i, \quad u_i = c_1(T_{si} - T_0) - \int_{p_0}^{p_1} p_1 d(1/\rho_1^0) + l_i + c_{p2}(T_i - T_{si}) + p_i(1/\rho_1^0 - 1/\rho_i^0), \tag{1.2}$$

$$l_i = l(p_i), \quad T_{si} = T_s(p_i) \quad (c_{p2} = c_2, \ c_1 = \text{const}).$$

In the absence of phase transitions the equations of state have the form

$$p_1 = p_0 + a_1^2(\rho_1^0 - \rho_{10}^0), \quad u_1 = c_1(T_1 - T_0) - \int_{p_0}^{p_1} p_1 d(1/\rho_1^0),$$
$$p_i = \rho_i^0 R_2 T_i, \quad u_i = c_2 T_i \quad (c_1, c_2 = \text{const}), \quad (i = 2, 3, \ldots, m). \tag{1.3}$$

We add the kinematic relations

$$\alpha_i = \rho_i/\rho_i^0, \quad \alpha_i = \frac{\pi}{6} n_i \delta_i^3, \quad \sum_{i=1}^m \alpha_i = 1.$$

We introduce the dimensionless variables

$$P_i = \frac{p_i}{p_0}, \ U_i = \frac{v_i}{a_*}, \ \Theta_i = \frac{T_i}{T_0}, \ \Theta_{si} = \frac{T_{si}}{T_0}, \ \Theta_{\sigma i} = \frac{T_{\sigma i}}{T_0} (a_*^2 = p_0/\rho_{10}^0),$$
$$\Phi_i = \frac{\rho_i}{\rho_{i0}}, \ \Phi_i^0 = \frac{\rho_i^0}{\rho_{i0}^0}, \ W_i = \frac{w_i}{a_*}, \ D_i = \frac{\delta_i}{\delta_{i0}}, \ N_i = \frac{n_i}{n_{i0}}, \tag{1.4}$$

the parameters

$$C_1 = \frac{c_1}{R_2}, \quad C_2 = \frac{c_2}{R_2}, \quad L = \frac{l}{R_2 T_0},$$

and the reduced variables

$$f_{mi}^* = \frac{f_{mi}}{\rho_{i0} a_*^2}, \quad f_{fi}^* = \frac{f_{fi}}{\rho_{i0} a_*^2}, \quad N_{ri}^* = \frac{8\nu_1 w_i}{\delta_{i0}^2 a_*^2},$$
$$q_i^{0*} = \frac{q_i^0}{\rho_{i0} a_* R_2 T_0}, \quad q_{0i}^* = \frac{q_{0i}}{\rho_{i0} a_* R_2 T_0}, \quad I_i^* = \frac{I_i}{\rho_{i0} a_*}, \quad \tau = a_* t. \tag{1.5}$$

Using (1.4) and (1.5), the system of equations (1.1) takes the following form in dimensionless variables:

$$\frac{\partial \Phi_1}{\partial \tau} + \frac{\partial U_1}{\partial x} = -\sum_{i=2}^m M_i I_i^*, \quad \frac{\partial \Phi_i}{\partial \tau} + \frac{\partial U_i}{\partial x} = I_i^*, \quad \frac{\partial N_i}{\partial \tau} + \frac{\partial U_i}{\partial x} = 0,$$
$$\frac{\partial U_1}{\partial \tau} = -\frac{\partial P_1}{\partial x} - \sum_{i=2}^m M_i(f_{mi}^* + f_{fi}^*), \quad \frac{\partial U_i}{\partial \tau} = -\frac{1}{r}\frac{\partial P_1}{\partial x} + f_{mi}^* + f_{fi}^*,$$

$$C_1\frac{\partial\Theta_1}{\partial\tau}=-\sum_{i=2}^{m}M_iq_i^{0*},\quad C_2\frac{\partial\Theta_i}{\partial\tau}+B\frac{\partial P_i}{\partial\tau}=-q_{0i}^*,\quad q_i^{0*}+q_{0i}^*=LI_i^*,$$
$$\frac{1}{2}\frac{\partial W_i}{\partial\tau}=\frac{P_i-P_1}{\delta_{i0}}-N_{ri}^*,\quad \frac{1}{2}\frac{\partial D_i}{\partial\tau}=\frac{W_i}{\delta_{i0}}+\frac{r}{6}I_i^*$$
$$\left(r_i=\rho_{i0}^0/\rho_{10}^0=r,\ M_i=\rho_{i0}/\rho_{10},\ B=\Theta_s'(C_1-C_2)-L'+r-1,\right.$$
$$\left.A_{10}^2=a_1^2/a_*^2\right). \tag{1.6}$$

The equations of state and Eqs. (1.3) take the form

$$\alpha_i=\alpha_{i_0}(\Phi_i-\Phi_i^0),\quad \alpha_i=\alpha_{i_0}(N_i+3D_i),$$
$$P_1=A_{10}^2\Phi_1^0,\quad P_i=\Phi_i^0+\Theta_i,\quad \sum_{i=1}^{m}\alpha_i=0\quad (i=2,3,\ldots,m). \tag{1.7}$$

For f_{mi}^*, f_{fi}^*, q_i^{o*}, q_{oi}^*, and I_i^* representing interphase interactions and for N_{ri}^* we have the expressions

$$f_{mi}^*=\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial\tau}(U_1-U_i),\quad f_{fi}^*=\frac{U_1-U_i}{\tau_{vi}},\quad N_{ri}^*=\frac{W_i}{\tau_{ri}},$$
$$q_i^{0*}=\frac{\Theta_1-\Theta_{\sigma i}}{\tau_i^0},\quad q_{0i}^*=\frac{\Theta_i-\Theta_{\sigma i}}{\tau_{0i}},\quad I_i^*=\frac{\Theta_{\sigma i}-\Theta_{si}}{\tau_{mi}},$$
$$\tau_{vi}=\frac{\pi r\delta_{i0}^2a_*}{6\chi\nu_1},\quad \tau_{ri}=\frac{\delta_{i0}^2a_*}{8\nu_1},\quad \tau_i^0=\frac{\rho_{i0}^0R_2\delta_{i0}^2a_*}{6\lambda_1\,\mathrm{Nu}_i^0},$$
$$\tau_{0i}=\frac{\rho_{i0}^0R_2\delta_{i0}^2a_*}{6\lambda_2\mathrm{Nu}_{0i}},\quad \tau_{mi}=\frac{\rho_{i0}^0T_0\delta_{i0}a_*}{6Fl_0}\quad (i=2,3,\ldots,m),$$

where τ_{vi}, τ_{ri}, τ_i^o, τ_{oi}, and τ_{mi} are the reduced relaxation times. We note that Eqs. (1.6) are written for the perturbations. We reduce this system to a more convenient form. Using Eqs. (1.7) we obtain from the equations for the conservation of mass and the number of bubbles

$$\frac{\alpha_{10}}{A_{10}^2}\frac{\partial P_1}{\partial\tau}+\sum_{i=2}^{m}\alpha_{i0}\left(r\frac{\partial\Phi_i^0}{\partial\tau}+3(r-1)\frac{\partial D_i}{\partial\tau}\right)+\sum_{i=1}^{m}\alpha_{i0}\frac{\partial U_i}{\partial x}=0,$$
$$\frac{\partial\Phi_i^0}{\partial\tau}+3\frac{\partial D_i}{\partial\tau}=I_i^*. \tag{1.8}$$

Instead of the momentum equation for the first phase we use the momentum equation for the whole mixture

$$\alpha_{10}\frac{\partial U_1}{\partial\tau}+\sum_{i=2}^{m}r\alpha_{i0}\frac{\partial U_i}{\partial\tau}+\frac{\partial P_1}{\partial x}=0.$$

Using (1.8) we write the second equation for pulsating motion in the form

$$\frac{W_i}{\delta_{i0}}=-\frac{1}{6}\left(r\frac{\partial\Phi_i^0}{\partial\tau}+3(r-1)\frac{\partial D_i}{\partial\tau}\right).$$

2. Propagation of Harmonic Vibrations. We consider the propagation of plane periodic waves. We seek the solution in the form of a damped traveling wave

$$\Phi_i,\ \Phi_i^0,\ P_i,\ U_i,\ \Theta_i \sim \exp[i(Kx-\omega t)] = e^{-dx}\exp[i(kx-\omega t)],$$
$$K = k + id,\ a_p = A_p a_* = \omega/k,$$

where K is the wave vector. The complex numbers d and a_p are, respectively, the damping factor and the phase velocity of the wave, determined by the imaginary and real parts of the wave vector. Henceforth instead of the frequency ω we use the dimensionless frequency $\eta = \omega\delta_0/a_*$, where δ_0 is some average diameter. After substituting into system (1.6) and (1.7), the condition for the existence of a nontrivial solution of this type is the vanishing of the determinant of the coefficients in the amplitudes of the perturbations. This condition gives a relation between the frequency of the perturbations and the wave vector. It is difficult to obtain the determinant directly in the polydisperse case, and therefore we find the necessary connection step by step by eliminating the amplitudes of the perturbations.

From the fourth of Eqs. (1.7)

$$U_i = \frac{\Pi_{v2i}U_1 + i\delta_0 KP_1}{\Pi_{v1i}}; \tag{2.1}$$

here

$$\Pi_{v1i} = \frac{r\delta_0}{\tau_{vi}} - i\eta(r+1/2),\quad \Pi_{v2i} = \frac{r\delta_0}{\tau_{vi}} - i\eta/2.$$

Substituting (2.1) into the third of Eqs. (1.6), we find

$$U_1 = \frac{\delta_0 K}{\eta}\pi_v P_1,\quad \pi_v = \left(1 + \sum_{i=2}^{m}\eta\frac{r\alpha_{i0}}{\Pi_{v1i}}\right)\left(\alpha_{10} + \sum_{i=2}^{m}\frac{r\alpha_{i0}\Pi_{v2i}}{\Pi_{v1i}}\right)^{-1}. \tag{2.2}$$

Hence

$$U_i = \frac{\delta_0 K}{\eta}\,\frac{\pi_v\Pi_{v2i} + i\eta}{\Pi_{v1i}}P_1. \tag{2.3}$$

From the remaining equations of system (1.6) we express $z_i = r\Phi_i^0 + 3(r-1)D_i$ in terms of P_1. From the equations for pulsating motion

$$\Pi_{ri}z_i = P_1 - P_i; \tag{2.4}$$

here

$$\Pi_{ri} = -\frac{1}{6}i\eta\frac{\delta_{i0}}{\delta_0}\left(i\eta\frac{\delta_{i0}}{2\delta_0} - \frac{\delta_{i0}}{\tau_{ri}}\right).$$

Taking account of (2.4) and the equation of state for the second phase (1.7) we obtain from the second equation for pulsating motion

$$\left(\frac{i\eta}{\delta_0}\frac{1}{\Pi_{ri}} - \frac{i\eta}{\delta_0} + (1-r)\frac{\Theta_s'}{\tau_{mi}}\right)P_i - \frac{i\eta}{\delta_0}\frac{1}{\Pi_{ri}}P_1 - (1-r)\frac{\Theta_{\sigma i}}{\tau_{mi}} + \frac{i\eta\Theta_i}{\sigma_0} = 0. \tag{2.5}$$

Using the equation for the internal energy of the bubbles and eliminating Θ_i we have from Eq. (2.5)

$$\begin{gathered}\left\{\left(\frac{1}{\tau_{0i}} - C_2\frac{i\eta}{\delta_0}\right)\left[\left(1-\Pi_{ri}^{-1}\right)\frac{i\eta}{\delta_0} - (1-r)\frac{\Theta_s'}{\tau_{mi}}\right] - B\frac{i\eta}{\delta_0}\frac{i\eta}{\delta_0}\right\}P_1 + \\ + \frac{i\eta}{\delta_0}\frac{1}{\Pi_{ri}}\left(\frac{1}{\tau_{0i}} - C_2\frac{i\eta}{\delta_0}\right)P_1 + \left[\left(\frac{1}{\tau_{0i}} - C_2\frac{i\eta}{\delta_0}\right)\frac{(r-1)}{\tau_{mi}} - \frac{i\eta}{\delta_0}\frac{1}{\tau_{0i}}\right]\Theta_{\sigma i} = 0.\end{gathered} \tag{2.6}$$

Eliminating Θ_i from the energy equations for the surface phase and for the bubbles, we obtain

$$\left[B\frac{i\eta}{\delta_0}\frac{1}{\tau_{0i}} + \left(\frac{1}{\tau_{0i}} - C_2\frac{i\eta}{\delta_0}\right)\frac{L\Theta_s'}{\tau_{mi}}\right]P_i + \left(\frac{1}{\tau_{0i}} - C_2\frac{i\eta}{\delta_0}\right)\frac{1}{\tau_i^0}\Theta_1 + \left[\frac{1}{\tau_{0i}^2} - \left(\frac{1}{\tau_{0i}} - C_2\frac{i\eta}{\delta_0}\right)\left(\frac{L}{\tau_{mi}} + \frac{1}{\tau_i^0} + \frac{1}{\tau_{0i}}\right)\right]\Theta_{\sigma i} = 0. \tag{2.7}$$

Eliminating P_i from (2.6) and (2.7), we have

$$\Theta_{\sigma i}=\left\{\frac{\Pi_{T3i}}{\Pi_{ri}}P_1+\left(1-i\eta C_2\frac{\tau_{0i}}{\delta_0}\right)\left[i\eta\left(\Pi_{ri}^{-1}-1\right)\frac{\tau_{mi}}{\delta_0}-\Theta_s'(1-r)+(i\eta)^2B\frac{\tau_{0i}}{\delta_0}\frac{\tau_{mi}}{\delta_0}\right]\Theta_1\right\}\left[\left(1-\Pi_{ri}^{-1}\right)\Pi_{T1i}+\Pi_{T2i}\right]^{-1},$$

where

$$\Pi_{T1i}=i\eta\frac{\tau_i^0}{\delta_0}\left[\left(L+\frac{\tau_{mi}}{\tau_i^0}+\frac{\tau_{mi}}{\tau_{0i}}\right)\left(1-i\eta C_2\frac{\tau_{0i}}{\delta_0}\right)-\frac{\tau_{mi}}{\tau_{0i}}\right];$$

$$\Pi_{T2i}=(r-1)\left[\Theta_s'-i\eta\left(\Theta_s'C_2\left(1+\frac{\tau_{0i}}{\tau_i^0}\right)\right)+B\right]\frac{\tau_i^0}{\delta_0}-i\eta\left[\Theta_s'L+i\eta B\left(L+\frac{\tau_{mi}}{\tau_{0i}}+\frac{\tau_{mi}}{\tau_i^0}\right)\frac{\tau_{0i}}{\delta_0}\right]\frac{\tau_i^0}{\delta_0};$$

$$\Pi_{T3i}=i\eta\left[\Theta_s'L\left(1-i\eta C_2\frac{\tau_{0i}}{\delta_0}\right)+i\eta B\frac{\tau_{mi}}{\delta_0}\right]\frac{\tau_i^0}{\delta_0}.$$

Substituting the expression found for $\Theta_{\sigma i}$ into the energy equation for the first phase we find

$$\Theta_1=\pi_T P_1; \tag{2.8}$$

here

$$\pi_T=\left\{\sum_{i=2}^{m}M_i\frac{\Pi_{T3i}}{\Pi_{T1i}}\left[1+\Pi_{ri}\left(1+\frac{\Pi_{T2i}}{\Pi_{T1i}}\right)\right]^{-1}\frac{\delta_0}{\tau_i^0}\right\}^{-1}\times$$

$$\times\left\{\sum_{i=2}^{m}M_i\frac{\Pi_{T5i}}{\Pi_{T1i}}\left[1+\Pi_{ri}\left(1+\frac{\Pi_{T6i}}{\Pi_{T5i}}\right)\right]\left[1+\Pi_{ri}\left(1+\frac{\Pi_{T2i}}{\Pi_{T1i}}\right)\right]^{-1}\frac{\delta_0}{\tau_i^0}-i\eta C_1\right\}^{-1};$$

$$\Pi_{T5i}=i\eta\left[\left(1-i\eta C_2\frac{\tau_{0i}}{\delta_0}\right)\left(L+\frac{\tau_{mi}}{\tau_{0i}}\right)-\frac{\tau_{mi}}{\tau_{0i}}\right]\frac{\tau_i^0}{\delta_0};$$

$$\Pi_{T6i}=(1-r)\,i\eta\left(\Theta_s'C_2+B\right)\frac{\tau_i^0}{\delta_0}-i\eta\left[\Theta_s'L+i\eta B\left(L+\frac{\tau_{mi}}{\tau_{0i}}\right)\frac{\tau_{0i}}{\delta_0}\right]\frac{\tau_i^0}{\delta_0}.$$

Eliminating $\Theta_{\sigma i}$ from Eqs. (2.6) and (2.7) and using (2.8) we obtain for z_i

$$z_i=\left(1+\frac{\pi_T\Pi_{T4i}}{\Pi_{T1i}+\Pi_{T2i}}\right)\left[\Pi_{ri}+\left(1+\frac{\Pi_{T2i}}{\Pi_{T1i}}\right)^{-1}\right]^{-1}P_1, \tag{2.9}$$

where

$$\Pi_{T4i}=(1+r)\left(1-i\eta C_2\frac{\tau_{0i}}{\delta_0}\right)-i\eta\frac{\tau_{mi}}{\delta_0}.$$

Finally we obtain the dispersion relation after substituting (2.2), (2.3), and (2.9) into the equation for the conservation of mass for the whole mixture and cancelling P_1:

$$\frac{\delta_0^2K^2}{\eta^2}=\left(\alpha_{10}\pi_v+\sum_{i=2}^{m}\alpha_{i0}y_i\right)^{-1}\left(\frac{\alpha_{10}}{A_{10}^2}+\sum_{i=2}^{m}\alpha_{i0}z_i\right); \tag{2.10}$$

here

$$y_i=(\pi_v\,\Pi_{v2i}+i\eta)/\Pi_{v1i}.$$

When m = 2, i.e., for the monodisperse case,

$$\frac{\delta_0^2 K^2}{\eta^2} = \left\{1 + \alpha_{20}(r-1)\left[1 - i\eta\left(\alpha_{10} + \frac{1}{2}\right)\frac{1}{r}\frac{\tau_v}{\delta_0}\right] \times \right.$$
$$\times \left[1 - i\eta\left(\frac{1}{2} + \alpha_{10}(\alpha_{20} - r\alpha_{10})\right)\frac{1}{r}\frac{\tau_v}{\delta_0}\right]^{-1}\right\} \times$$
$$\times \left\{\frac{\alpha_{10}}{A_{10}^2} + \alpha_{20}\left[\left(1 + \frac{\Pi_0 + \pi\Pi_3 - \Pi_1\Pi_4}{\Pi_2\Pi_4 + \Pi_3}\right)^{-1} + i\eta\frac{1}{6}\left(i\eta\frac{1}{2} - \frac{\delta_0}{\tau_r}\right)\right]^{-1}\right\}; \qquad (2.11)$$

here

$$\Pi_0 = \mathrm{Re}\,\Pi_0 + \mathrm{Im}\,\Pi_0, \quad \mathrm{Re}\,\Pi_0 = (1-r)\left(\Theta_s'(C_2 - C_1/M) + B\right),$$
$$\mathrm{Im}\,\Pi_0 = (r-1)\left(\Theta_s' C_2\left(1 + \frac{\tau_0}{\tau^0}\right) + B\right)\eta\frac{C_1}{M}\frac{\tau^0}{\delta_0}, \quad \pi = \Theta_s' L\tau_0/\tau_m,$$
$$\Pi_1 = \Theta_s' L + i\eta B\frac{\tau_m}{\delta_0}, \quad \Pi_2 = -i\eta C_2\frac{\tau_m}{\delta_0},$$
$$\Pi_3 = i\eta\left(L + \frac{\tau_m}{\tau^0}\right)\frac{C_1}{M}\frac{\tau^0}{\delta_0}, \quad \Pi_4 = \left(1 + L\frac{\tau_0}{\tau_m}\right) - i\eta\left(1 + \frac{\tau_0}{\tau^0} + L\frac{\tau_0}{\tau_m}\right)\frac{C_1}{M}\frac{\tau^0}{\delta_0}.$$

Equation (2.10) is written in the most general form. Special forms can be obtained from this equation. For example, $\pi_T = 0$ corresponds to the case $C_1 = \infty$; i.e., the carrier phase behaves as a thermostat. The equation $z_i = 0$ implies that phase transitions, heat transfer, and inertial and dissipative effects for radial pulsations do not lead to dispersion.

For the dimensionless phase velocity of sound $A_p = \delta_0 k/\eta$ and the damping factor d we have

$$A_p = \left[\frac{2}{\varphi + (\varphi^2 + \psi^2)^{1/2}}\right]^{1/2}, \quad d = \frac{1}{2}\eta A_p \psi/\delta_0,$$

where φ and ψ are the real and imaginary parts of the right-hand side of Eq. (2.10). As $\eta \to 0$ and $\eta \to \infty$ we find, respectively, the equilibrium and "frozen" sound speeds

$$a_e^{-2} = a_*^{-2}(1 + \alpha_{20}(r-1))\left(\frac{\alpha_{10}}{A_{10}^2} + \frac{\alpha_{20}}{L}(1-r)\left(\Theta_s'(C_2 + C_1/M) + B + (1 - \Theta_s')L\right)\right),$$
$$a_f^{-2} = a_1^{-2}\alpha_{10}\frac{1 - \alpha_{20}(r-1)}{1 + 2\alpha_{10}(\alpha_{20} - r\alpha_{10})} \simeq a_1^{-2}.$$

Here $\alpha_{20} = 1 - \alpha_{10}$; i.e., α_{20} is the total volume concentration of the gas phase.

For $p_0 = 10$ bar we have for water

$$\Theta_s' \simeq 10^{-1}, \ L \simeq 10, \ C_1 \simeq 2, \ C_2 \simeq 10, \ B \simeq -1,$$
$$r \simeq 2\cdot 10^{-2}, \ A_{10}^{-2} \simeq 10^{-3}, \ \alpha_{20} \simeq 10^{-2};$$

hence $\Theta_s' C_1/M \gg \Theta_s' C_2$, B, L $(\Theta_s' - 1)$, and therefore

$$a_e^2 = a_*^2 Lr/\Theta_s' C_1; \qquad (2.12)$$

i.e., in this case the equilibrium sound speed is practically independent of the volume concentration of bubbles. The equilibrium sound speed (2.12) agrees with the speed from [12]. For the parameters listed, $a_e \approx 10$ m/sec.

When the inertia of the bubbles resulting from their mass can be neglected, i.e., for $r \ll 1$, we obtain for the "frozen" damping factor d_f:

$$d_f = 2\alpha_{10}^2 A_{10}^{-1}\left(\sum_{i=2}^{m}\alpha_{i0}/\tau_{vi}^0\right)(1 + 2\alpha_{10}(1 - \alpha_{10}))^{-3/2} \quad (\tau_{vi}^0 = \tau_{vi}/r).$$

In the monodisperse case for $\alpha_{20} \ll 1$

$$d_f = 2\alpha_{20}/A_{10}\tau_v^0.$$

It should be noted that $d_f = 0$ for the one-velocity model. This results from the fact that as the frequency approaches infinity the radial motion of the bubbles is "frozen," i.e., $D_i = 0$, and therefore the dissipation in radial pulsations because of heat transfer and viscosity vanishes. From Eq. (2.3), when $r \ll 1$, we have for the asymptotic form of the ratio of the perturbations of the i-th ($i = 2, 3, \ldots, m$) and the first phase as $\eta \to \infty$

$$U_i/U_1 \simeq (1 + 2\alpha_{10}) \simeq 3.$$

Thus, according to the m-velocity model, dissipation because of viscosity remains as a consequence of phase transitions.

3. No Phase Transitions. In this case we have from the equations of state (1.3)

$$\Pi_{T1i} = 1 - i\eta C_2\left(1 + \frac{\tau_i^0}{\tau_{0i}}\right)\frac{\tau_{0i}}{\delta_0},\ \Pi_{T2i} = i\eta B\left(1 + \frac{\tau_i^0}{\tau_{0i}}\right)\frac{\tau_{0i}}{\delta_0},$$

$$\Pi_{T3i} = -\Pi_{T6i} = i\eta B\,\frac{\tau_i^0}{\delta_0},\ \Pi_{T4i} = 1,\ \Pi_{T5i} = -\,i\eta C_2\frac{\tau_i^0}{\delta_0}.$$

For a monodisperse mixture we have the dispersion relation

$$\frac{\delta_0^2 K^2}{\eta^2} = \left\{1 + \alpha_{20}(r-1)\left[1 - i\eta(\alpha_{10} + 1/2)\frac{1}{r}\frac{\tau_v}{\delta_0}\right]\left[1 - i\eta\left(\frac{1}{2} + \right.\right.\right.$$

$$\left.\left.+\,\alpha_{10}(\alpha_{20} - r\alpha_{10})\right)\frac{1}{r}\frac{\tau_v}{\delta_0}\right]^{-1}\Bigg\}\left\{\frac{\alpha_{10}}{A_{10}^2} + \alpha_{20}\left[\left(1 + (1 - i\eta B\left(1 + \frac{\tau^0}{\tau_0}\right)\times\right.\right.\right.$$

$$\left.\left.\times\,\frac{C_1}{M}\frac{\tau_0}{\delta_0}\right)\left(C_1/M + C_2\left(1 - i\eta\left(1 + \frac{\tau^0}{\tau_0}\right)\frac{C_1}{M}\frac{\tau_0}{\delta_0}\right)\right)^{-1}\right)^{-1} + i\eta\frac{1}{6}\left(i\eta/2 - \frac{\delta_0}{\tau_r}\right)\Bigg]^{-1}\Bigg\},$$

where

$$B = -1,\ C_2 = \gamma_2/(\gamma_2 - 1)\ (\gamma_2 = c_{p2}/c_{v2}).$$

For the equilibrium sound speed we have the expression

$$a_e^{-2} = a_*^{-2}(1 + \alpha_{20}(r-1))\cdot\left(\frac{\alpha_{10}}{A_{10}^2} + \alpha_{20}\left(1 + \frac{1-\gamma_2}{\gamma_2 + (\gamma_2 - 1)C_1/M}\right)\right).$$

For a pressure $p_0 \sim$ 1-10 bar and for volume concentrations $\alpha_{20} \sim 10^{-2}$ in the liquid-bubble mixture for

$$r \simeq 10^{-3},\ C_1/M \simeq 10^5,\ A_{10}^{-2} \simeq 10^{-3},$$

we have

$$a_e^2 = \frac{a_*^2}{\alpha_{10}\alpha_{20}} = \frac{p_0}{\rho_{10}^0\alpha_{10}\alpha_{20}}.$$

The expressions obtained for the equilibrium and "frozen" sound speeds agree with those which follow from a consideration of the conditions for the existence of condensation waves [13] in two-phase media.

4. Generalization for a Continuous Distribution of Bubble Sizes. So far it has been assumed everywhere that the bubble sizes have a discrete distribution, i.e., there are $m-1$ kinds of bubbles. All this can be generalized for a continuous distribution.

We introduce a density distribution function for the bubble diameters $\varkappa(\delta_0')$ such that

$$1 - \alpha_{10} = \int_{\delta_{10}}^{\delta_{20}} \varkappa(\delta_0')\,d\delta_0' = \lim \sum_i \varkappa(\delta_{i0}')\,\Delta\delta_{i0}'.$$

Each term $\varkappa(\delta_{i0}')\Delta\delta_{i0}'$ corresponds to an α_{i0} of the previous treatment. Therefore, formally replacing α_{i0} by $\varkappa(\delta_{i0}')\Delta\delta_{i0}'$ and going to the limit as $\max(\Delta\delta_{i0}') \to 0$, we have instead of the dispersion relation (2.11)

$$\frac{\delta_0^2 K^2}{\eta^2}\left[\frac{\alpha_{10}}{A_{10}^2}+(1-\alpha_{10})\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^* z\, d\delta_0'\right]\left[\alpha_{10}\pi_v+(1-\alpha_{10})\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^* y\, d\delta_0'\right]^{-1}, \tag{4.1}$$

where $\varkappa^*$ is the normalized distribution density; i.e., $\varkappa = (1-\alpha_{10})\varkappa^*$.

For the remaining variables in (4.1) we have

$$\pi_v=\left(1/\alpha_{10}+\eta M\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^*\,\Pi_{v1}^{-1}d\delta_0'\right)\left(1+M\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^*\Pi_{v2}\Pi_{v1}^{-1}d\delta_0'\right)^{-1};$$

$$y=(\pi_v\Pi_{v2}+i\eta)/\Pi_{v1},\quad z=\left(1+\frac{\pi_T\Pi_{T4}}{\Pi_{T1}+\Pi_{T2}}\right)\left[\Pi_r+\left(1+\frac{\Pi_{T2}}{\Pi_{T1}}\right)^{-1}\right]^{-1},$$

$$\pi_T=\left\{\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^*\frac{\Pi_{T3}}{\Pi_{T1}}\frac{\delta_0'}{\tau^0}\left[1+\Pi_r\left(1+\frac{\Pi_{T2}}{\Pi_{T1}}\right)\right]^{-1}d\delta_0'\right\}\left\{\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^*\frac{\Pi_{T5}}{\Pi_{T1}}\frac{\delta_0'}{\tau^0}\left[1+\right.\right.$$

$$\left.\left.+\Pi_r\left(1+\frac{\Pi_{T6}}{\Pi_{T1}}\right)\right]\left[1+\Pi_r\left(1+\frac{\Pi_{T2}}{\Pi_{T1}}\right)\right]^{-1}d\delta_0'-i\eta C_1/M\right\}^{-1}$$

$$(M=(1-\alpha_{10})\,r/\alpha_{10}).$$

The expressions for Π_T, Π_v, and Π_r are the same as before with δ_{i0} replaced by δ_0'. Equation (4.1) agrees with that for a monodisperse mixture if the density distribution function is taken as the delta function $e(\delta_0'-\delta_0)$. In the one-velocity case for isothermal behavior of the bubbles the dispersion relation takes the form

$$\frac{\delta_0^2K^2}{\eta^2}=\alpha_{10}\left\{\frac{\alpha_{10}}{A_{10}^2}+(1-\alpha_{10})\int\limits_{\delta_{10}}^{\delta_{20}}\varkappa^*\left[1-\frac{\eta^2}{12}\left(\frac{\delta_0'}{\delta_0}\right)^2\right]^{-1}d\delta_0'\right\}.$$

This equation agrees with the one obtained in [3]. Heat transfer is taken into account in a similar way [13].

5. Results of Calculations. In the monodisperse case the effects of various factors on the phase velocity and the damping of sound were tested. Two-velocity effects were checked by using the Levich and Stokes equations for the force of friction.

Figures 1 and 2 show the dispersion curves for the following values of the thermodynamic parameters:

$$\rho_{10}^0=10^3\ \text{kg/m}^3,\quad a_1=1.5\cdot10^3\,\text{m/sec},\ \nu_1=0.2\cdot10^{-6}\ \text{m}^2/\text{sec},$$
$$c_1=4.4\cdot10^3\ \text{m}^2/\text{sec}^2\cdot\text{deg},\ \gamma_2=1.4,\ \lambda_2=2.47\cdot10^{-2}\ \text{kg}\cdot\text{m/sec}^3\cdot\text{deg},$$
$$\lambda_1=0.65\,\text{kg}\cdot\text{m/sec}^3\cdot\text{deg},\ p_0=2\ \text{bar},\ T_0=300\ {}^\circ\text{K},\ R_2=0.287\cdot10^3\ \text{m}^2/\text{sec}^2\cdot\text{deg}.$$

Curve 1 was calculated from the dispersion relation without phase transitions for the discrete distribution of bubbles given in [1] with $Nu_i^0 = 10^2$ and $Nu_{0i} = 10$, curve 2 from [1] for the same distribution, 3 for an average monodisperse mixture ($\delta_0 = 0.11\cdot10^{-3}$ m), and 4 shows the spread of experimental points.

Figure 3 shows the dispersion curves for values of the thermodynamic parameters for water:

$$p_0=10\ \text{bar},\ T_0=452\ {}^\circ\text{K},\ l=2.014\cdot10^6\text{m}^2/\text{sec}^2,\ a_1=1.5\cdot10^3\ \text{m/sec},$$
$$c_1=4.4\cdot10^3\ \text{m}^2/\text{sec}^2\cdot\text{deg},\ \lambda_1=0.68\,\text{kg}\cdot\text{m/sec}\cdot\text{deg},\ \nu_1=0.2\cdot10^{-6}\text{m}^2/\text{sec},$$
$$R_2=0.59\cdot10^3\text{m}^2/\text{sec}^2\cdot\text{deg},\ c_{p2}=1.9\cdot10^3\,\text{m}^2/\text{sec}^2\cdot\text{deg},\ \lambda_2=$$
$$=3.14\cdot10^{-2}\ \text{kg}\cdot\text{m/sec}^3\cdot\text{deg}.$$

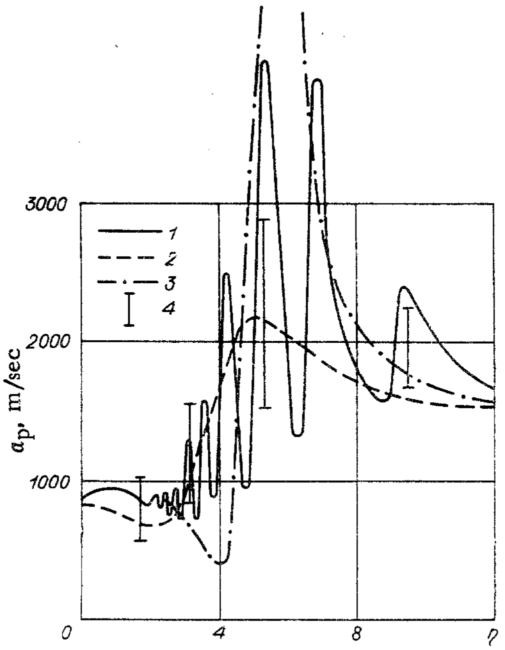

Fig. 1

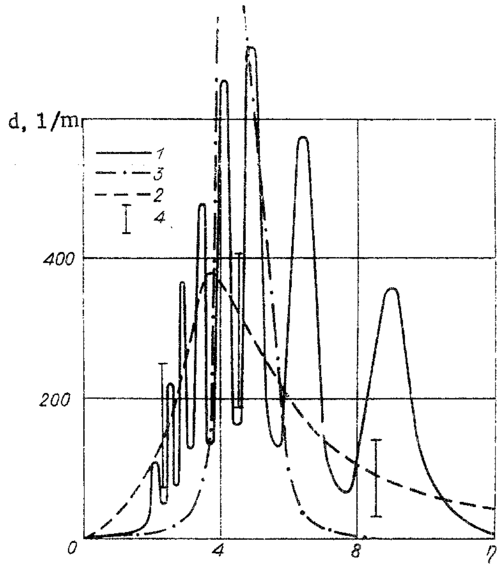

Fig. 2

The Nusselt numbers were taken as $Nu_i^0 = 3 \cdot 10^2$ and $Nu_{0i} = 30$.

Figure 3 shows the dispersion curves in the monodisperse case for bubbles with $\delta_0 = 10^{-3}$ m and cubic content $\alpha_{20} = 0.2 \cdot 10^{-3}$. Curve 1 is for "equilibrium" phase transitions, and 2 and 3 are dispersion curves with kinetics F corresponding to $F = 10^{-2}$ and 10^{-4} kg·deg·sec/m^4, respectively. The dispersion curves for $F \gtrsim 1$ kg·deg·sec/m^4 practically coincide with the dispersion curves for "equilibrium" phase transitions, and those for $F \lesssim 10^{-4}$ kg·deg·sec/m^4 with the curves for no phase transitions.

Calculations showed that in taking account of the actual heat transfer, multivelocity effects produced practically no change in the phase velocity and the damping. An increase in the volume concentration of bubbles of fixed diameter leads to a displacement of the extremes of the phase velocity and the damping factor toward higher frequencies of the perturbations.

For bubbles of diameter $\delta_0 \sim 10^{-4}$–10^{-3} m heat transfer is determined by their thermal resistance, since $\tau^0/\tau_0 \sim O(10^2)$. For a change in τ^0, i.e., in Nu^0, by an order of magnitude, the dispersion curves are unchanged, but for a change in τ_0, i.e., in Nu_0, by an order of magnitude, the damping factor is changed appreciably.

For realistic values of Nu_0 the damping factor for no phase transitions varies nonmonotonically with the Nusselt number Nu_0. As the Nusselt number Nu_0 is increased the damping factor increases to a maximum and then gradually decreases.

Calculations of the dispersion relations showed that in all the cases considered π_T could be set equal to zero; i.e., the liquid could be considered a thermostat. For phase transitions with kinetics F an increase in F leads to an increase in the damping factor. The dispersion curves with kinetics F in this case approach the dispersion curve with "equilibrium" phase transitions. Effects due to a change of the heat-transfer coefficient were tested when phase transitions occur. In contrast with the case of no phase transitions the dispersion curves for "equilibrium" phase transitions are more strongly dependent on external heat transfer than on internal.

The introduction of polydispersity leads to an essentially nonmonotonic dependence of the dispersion curves on the frequency of the perturbations. This accounts for the large spread of experimental points, particularly in [1], where dispersion curves in monodisperse and polydisperse cases are compared with experiment. Two size distributions of bubbles were assumed: first by the introduction of a certain average bubble radius and second by the introduction of polydispersity. The dispersion relations in [1] agree with ours in form if equality of phase velocities and isothermal behavior of the bubbles are assumed, i.e., there is dissipation only from viscosity in radial pulsations. Dissipation from acoustic radiation

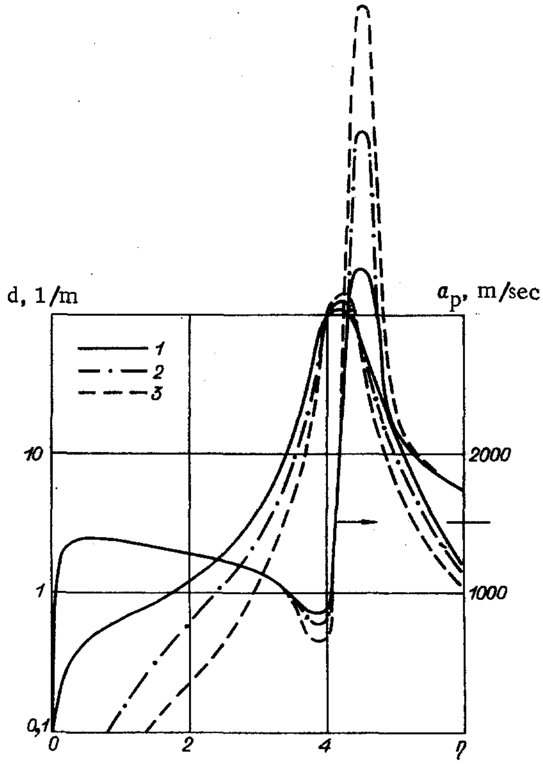

Fig. 3

and heat transfer was taken into account in [1] by introducing an effective viscosity. We make a different but equivalent assumption that the viscosity $\nu_1 = O(10^{-2})$ m²/sec. As noted in [4] this value of the effective viscosity is more than five times larger than it should be actually because of heat transfer, acoustic radiation, and viscosity for radial pulsations.

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